

Analogues of the Smale and Hirsch Theorems for Cooperative Boolean and Other Discrete Systems

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Dedicated to Avner Friedman, on the occasion of his 75th birthday.

Abstract

Discrete dynamical systems defined on the state space $\Pi = \{0, 1, \dots, p-1\}^n$ have been used in multiple applications, most recently for the modeling of gene and protein networks. In this paper we study to what extent well-known theorems by Smale and Hirsch, which form part of the theory of (continuous) monotone dynamical systems, generalize or fail to do so in the discrete case.

We show that that arbitrary m -dimensional systems cannot necessarily be embedded into n -dimensional cooperative systems for $n = m+1$, as in the Smale theorem for the continuous case, but we show that this is possible for $n = m+2$ as long as p is sufficiently large.

We also prove that a natural discrete analogue of strong cooperativity implies non-trivial bounds on the lengths of periodic orbits and imposes a condition akin to Lyapunov stability on all attractors. Finally, we explore several natural candidates for definitions of irreducibility of a discrete system. While some of these notions imply the strong cooperativity of a given cooperative system and impose even tighter bounds on the lengths of periodic orbits than strong cooperativity alone, other plausible definitions allow the existence of exponentially long periodic orbits.

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1 Introduction

Let (L, \leq) be a linearly ordered set, let $n \geq 1$, let $L_1, \dots, L_n \subseteq L$ with the induced order, and consider the set $\Pi = \prod_{i=1}^n L_i$. A map $g : \Pi \rightarrow \Pi$ defines the discrete dynamical system

$$x(t+1) = g(x(t)), \quad x(t) \in \Pi, \quad (1)$$

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We call (1) an *n-dimensional, discrete system* and also identify it with the pair (Π, g) . For most of this paper, L will be the set of real numbers with the natural order, and $L_i = \{0, \dots, p-1\}$ for some fixed integer $p > 1$. In this case we speak of an *n-dimensional, p-discrete system*. The case $p = 2$ corresponds to the so-called *Boolean networks* or *Boolean systems* which are used in various disciplines, notably in the study of gene regulatory systems [10, 11, 13, 16, 17, 18, 23, 24]. If all L_i 's are finite, then we may without loss of generality assume that $L_i = \{0, \dots, p_i-1\}$ for some $p_i > 1$, but the p_i 's are not necessarily all equal. In this case we speak of a *finite discrete system*.

Define a partial order on Π by $x \leq y$ if $x_i \leq y_i$ for $i = 1, \dots, n$. We call this relation the *cooperative order*, and we will not make a notational distinction between it and the order relation on L . A discrete system (1) is said to be *cooperative* if $x(0) \leq y(0)$ implies $x(t) \leq y(t)$ for every $t \geq 0$, where $x(t), y(t)$ are the solutions of the system with initial conditions $x(0), y(0)$ respectively. Clearly this is equivalent to the property that $x \leq y$ implies $g(x) \leq g(y)$. Discrete cooperative systems have been proposed as a tool to study genetic networks by Sontag and others [22, 23].

The cooperativity property has a well-studied counterpart in continuous dynamical systems

$$\frac{dx_i}{dt} = f_i(x), \quad i = 1 \dots n, \quad (2)$$

for C^1 vector fields $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Namely, the system (2) is *cooperative* if whenever $x(t), y(t)$ are two solutions such that $x_i(0) \leq y_i(0)$, $i = 1, \dots, n$, then $x_i(t) \leq y_i(t)$ for every $t > 0$, $i = 1, \dots, n$. Cooperative systems are canonical examples of so-called monotone systems, which have been studied extensively by M. Hirsch, H. Smith, H. Matano, P. Poláčik and others, and more recently by Sontag and collaborators in the context of gene regulatory networks under exclusively positive feedback [2, 7, 9, 14, 20].

The Smale and Hirsch Theorems

In the present paper we consider two important results from the theory of (continuous) monotone dynamical systems, and we show to what extent these results either generalize or fail to do so in the context of cooperative discrete systems (1).

The first result was originally published by S. Smale in the 1970's [19]. It states in this context that any compactly supported, $(n-1)$ -dimensional, C^1 dynamical system defined on $H = \{x \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0\}$ can be embedded into some cooperative C^1 system (2). Equivalently, the dynamics of cooperative systems can be completely arbitrary on unordered hyperplanes such as H . See also [5], where the cooperative system (2) is shown to have bounded solutions and only two equilibria outside of H .

One way to regard the Smale theorem in the discrete case would be to ask whether discrete cooperative systems can have arbitrary dynamics on unordered sets $H = \{x \in \Pi \mid x_1 + \dots + x_n = \text{const.}\}$. This is trivially true; see Lemma 3.

An alternative approach is to study whether one can embed an arbitrary m -dimensional p -discrete system (1) into a cooperative $(m+1)$ -dimensional p -discrete system. We show that the answer to this question is *no* (Theorem 8, item 3), but that the statement is true (for sufficiently large p) if ' $m+1$ ' is weakened to ' $m+2$ ' (Theorem 8, item 2).

The second result for continuous cooperative systems was proved by M. Hirsch in [7]. A continuous cooperative system is *strongly cooperative* if for every two different initial conditions $x(0) \leq y(0)$ we have $x_i(t) < y_i(t)$ for all $i = 1, \dots, n$ and $t > 0$. A closely related definition involves the digraph G associated with the system: in the cooperative case, G is defined as having nodes $1, \dots, n$, and an arc from i to j is present if and only if $\partial f_j / \partial x_i(x) > 0$ on \mathbb{R}^n . A continuous cooperative system (2) is strongly cooperative if the digraph G is strongly connected [20]; we refer to the latter property as the *irreducibility* of the system (2). Hirsch's theorem states that almost every bounded solution of a strongly cooperative system (2) converges towards the set of equilibria. This result rules out stable periodic orbits and chaotic behavior. It was also generalized for abstract order relations in Banach spaces by Hirsch and extended to continuous-space, discrete-time maps by Tereščák, Poláčik and collaborators; see [8, 14, 15, 6].

For finite discrete systems, we will consider analog definitions of strong cooperativity and of irreducibility of a cooperative system (1). We are particularly interested in whether these definitions rule out the existence of exponentially long periodic orbits, which in finite discrete systems can be considered analogues of chaotic attractors. We show that strong cooperativity does not rule out periodic orbits altogether, but that it puts a non-trivial, subexponential bound on their lengths and imposes a condition akin to Lyapunov stability on all attractors. Finally, we explore several natural candidates for definitions of irreducibility of a finite discrete system. We show that predicted properties of the system can dramatically change when subtle changes to our definitions are made. While some plausible definitions of irreducibility still allow for exponentially long periodic orbits (and hence do not imply strong cooperativity), other definitions of irreducible cooperative systems imply strong cooperativity and impose a bound of n (the dimension of the system) on the lengths of periodic orbits. This is a much tighter bound than the one implied by strong cooperativity alone.

Outline of the Sections

In Section 2 we give a general condition under which an arbitrary m -dimensional, p -discrete system can be embedded into a cooperative n -dimensional, q -discrete system (Proposition 4). We rely on several standard results from the literature, especially a generalization of the classical Sperner theorem. In Section 3 we provide bounds on the maximum size $d_{n,p}$ of an unordered subset of Π , and we use these bounds to study the special cases $n = m + 1$ and $n = m + 2$ (Theorem 8). In Section 4 we prove a general result on extensions of cooperative partial functions on Π to cooperative systems on Π and discuss how our results are related to a certain generalization of Smale's theorem. We give a short discussion in Section 5 about applying Theorem 8 to the case of *almost cooperative* discrete systems [23], by showing a simple example of an almost cooperative Boolean system of dimension m that cannot be embedded into a cooperative Boolean system of dimension $m + 1$. In Section 6 we introduce a counterpart of strong cooperativity for finite discrete systems (1) and show that it imposes substantial restrictions on the possible dynamics. In particular, we show that strongly cooperative p -discrete systems cannot have exponentially long periodic orbits. In Section 7 we explore several natural definitions of irreducibility for finite discrete systems and prove bounds on the lengths of periodic orbits in cooperative systems that are

irreducible in the sense of these definitions.

2 Unordered Sets and Cooperative Embeddings of p -discrete systems

Let $\Sigma := \prod_{i=1}^m L_i$ and $\Pi = \prod_{i=1}^n L_i^*$ and consider an arbitrary map $f : \Sigma \rightarrow \Sigma$. A *cooperative embedding* of (Σ, f) into a cooperative system (Π, g) as in (1) is an injective function $\phi : \Sigma \rightarrow \Pi$ such that $g(\phi(x)) = \phi(f(x))$ for every $x \in \Sigma$. If $\Pi = \prod_{i=1}^n \{0, \dots, p_i - 1\}$, then we define $S(x) = x_1 + \dots + x_n$ for $x \in \Pi$. These definitions will be used throughout this paper.

For the remainder of this section and the next one, let $\Pi = \{0, \dots, p - 1\}^n$ for some fixed integer $p > 1$. We will compute the least dimension n such that any m -dimensional p -discrete system (Σ, f) can be embedded into an n -dimensional cooperative system (Π, g) .

A subset $A \subseteq \Pi$ is said to be *unordered* if no two different elements $a, b \in A$ satisfy $a \leq b$. Define the set

$$D := \{x \in \Pi \mid S(x) = \lfloor n(p-1)/2 \rfloor\}, \quad d_{n,p} := |D|. \quad (3)$$

This set D is clearly unordered, because if $x \leq y$, $x \neq y$, then necessarily $S(x) < S(y)$, and x, y cannot be both in D . Notice that $d_{n,2} = \binom{n}{\lfloor n/2 \rfloor}$. We quote a generalization of Sperner's Theorem [1, 3], which states that D is a set of maximum size in Π with this property:

Lemma 1 *Consider the set $\Pi = \{0, 1, \dots, p - 1\}^n$, under the cooperative order \leq . Then $|A| \leq d_{n,p}$, for any unordered set A .*

The following lemma will be important below, see Proposition 5.2 in [9] for a proof.

Lemma 2 *Consider a cooperative map g defined on a space Π . Then any periodic orbit is unordered.*

Another basic property of unordered sets is the following ‘trivial embedding’ result, which is well known at least for the Boolean case (see for instance [23]).

Lemma 3 *Let $A \subseteq \Pi$ be unordered, and let $\gamma : A \rightarrow A$ be an arbitrary function. Then there exists a cooperative system (1) such that $g|_A = \gamma$.*

Proof. Let \hat{A} be any unordered subset of Π which contains A , and which is maximal with respect to this property. Define $g(a) := \gamma(a)$ for $a \in A$, and $g(a) = a$ for $a \in \hat{A} - A$. For all other $x \in \Pi$, there must exist $a \in \hat{A}$ such that either $a \leq x$ or $x \leq a$, by the maximality of \hat{A} . If $x \leq a$ let $g(x) := [0, \dots, 0]$, and if $a \leq x$ let $g(x) := [p - 1, \dots, p - 1]$. ■

Proposition 4 *Let n, m be positive integers, $p, q > 1$, and $\Pi = \{0, 1, \dots, p - 1\}^n$, $\Sigma = \{0, 1, \dots, q - 1\}^m$. Then the following are equivalent:*

(i) *Any discrete system (Σ, f) can be embedded into a cooperative discrete system (Π, g) .*

(ii) $q^m \leq d_{n,p}$.

Proof. Suppose first that $q^m \leq d_{n,p}$, and consider any discrete system (Σ, f) . We use an arbitrary injective function $\phi : \Sigma \rightarrow \Pi$ such that $A := \text{Im}(\phi) \subseteq D$. Let $\gamma(y) := \phi(f(x))$ whenever $y \in A$, where $x = \phi^{-1}(y)$. Thus by construction $\gamma(\phi(x)) = \gamma(y) = \phi(f(x))$ holds for $x \in \Sigma$. Apply Lemma 3 to define g and obtain a full cooperative embedding.

Now assume (i) in the statement. To prove that (ii) must hold, simply consider a map f on Σ which generates a single orbit with period q^m . By (i), there exists an embedding into Π , and the image of Σ is unordered in Π by Lemma 2. The inequality follows from Lemma 1. \blacksquare

Another form of cooperative embedding was given by Smith [21] for a large class of non-cooperative, but possibly continuous maps. In that case $n = 2m$ holds. By Proposition 4, a much sharper bound holds for the discrete case.

3 Bounds on Discrete Cooperative Embeddings

Let $p > 1$, $n > 0$ be arbitrary, and let $\Pi, D, d_{n,p}$ be as in the previous section. We begin this section with several lemmas.

Lemma 5 $d_{n,p} \geq \frac{p^{n-1}}{n}$.

Proof. Let $S_j := \{x \in \Pi \mid S(x) = j\}$, for $j = 0, \dots, n(p-1)$. Each of these sets is unordered, and therefore $|S_j| \leq d_{n,p}$ by Lemma 1. Therefore

$$p^n = \sum_{j=0}^{n(p-1)} |S_j| \leq (n(p-1) + 1)d_{n,p} \leq npd_{n,p}.$$

\blacksquare

Lemma 6 Let c be such that $0 < c < p$. Then $d_{n,p} \geq c^n$, for all sufficiently large n .

Proof. By Lemma 5, it is sufficient to show that $p^{n-1}/n \geq c^n$. But this is equivalent to $\ln p \geq \ln c + (\ln n + \ln p)/n$. This inequality holds for large n since $\ln p > \ln c$. \blacksquare

We now prove an upper bound for $d_{n,p}$.

Lemma 7 Let $p, n > 1$. Then $d_{n+1,p} < p^n$.

Proof. Let p, n be as in the assumptions, and let x be a randomly chosen element of $\{0, \dots, p-1\}^{n+1}$ with the uniform distribution. For x to be in D , we must have $\lfloor n(p-1)/2 \rfloor - p + 1 \leq x_1 + \dots + x_{n-1} \leq \lfloor n(p-1)/2 \rfloor$ and $x_n = x_1 + \dots + x_{n-1} - \lfloor n(p-1)/2 \rfloor$. Let A be the event that $\lfloor n(p-1)/2 \rfloor - p + 1 \leq x_1 + \dots + x_{n-1} \leq \lfloor n(p-1)/2 \rfloor$. Our assumption on n implies that $P(A) < 1$. Moreover, note that $P(x_n = x_1 + \dots + x_{n-1} - \lfloor n(p-1)/2 \rfloor \mid A) = \frac{1}{p}$. Thus $P(x \in D) = \frac{d_{n+1,p}}{p^{n+1}} < \frac{1}{p}$, and the lemma follows. \blacksquare

The above estimates have important consequences for embeddings of m -dimensional finite discrete systems into n -dimensional cooperative finite discrete systems. In particular, unlike for continuous systems, for large m , an m -dimensional p -discrete system can in general not be embedded into an $(m + 1)$ -dimensional p -discrete cooperative system.

Theorem 8 *The following statements hold:*

1. *For every $p > 1$, and for every $m > 0$, there exists $n > m$ such that every m -dimensional p -discrete system can be embedded into an n -dimensional cooperative p -discrete system.*
2. *For every $m > 0$, there exists p_0 such that for every $p > p_0$ every m -dimensional p -discrete system can be embedded into a cooperative p -discrete system of dimension $m + 2$.*
3. *For every $m, p > 1$ there exists an m -dimensional p -discrete system that cannot be embedded into a cooperative p -discrete system of dimension $m + 1$.*

Proof. The first two statements are immediate consequences of Lemma 5 and Proposition 4. For the first one, let n be large enough so that $p^m \leq p^{n-1}/n$. Then $p^m \leq d_{n,p}$, and the conclusion follows. For the second statement, let simply $p \geq m + 2$. Then

$$p^m \leq \frac{p^{m+1}}{m+2} = \frac{p^{(m+2)-1}}{m+2} \leq d_{m+2,p}.$$

For the third statement, let $m, p > 1$. Let f be defined on Π so as to generate a single orbit of length p^m . Then the image of Π under any embedding ϕ into $\Sigma = \{0, \dots, p-1\}^{m+1}$ would also generate a periodic orbit of this length. Assuming that the system defined on Σ is cooperative, the set $\text{Im}(\phi)$ must be unordered by Lemma 2, and therefore $p^m \leq d_{m+1,p}$ by Lemma 1. But by Lemma 7, $d_{m+1,p} < p^m$, a contradiction. ■

Note that we are restricting our attention to the case where $p = q$, i.e. both systems have the same level of discretization. This is relevant for instance in the special case of Boolean networks. But if we allow $q \neq p$, then the analogue of Theorem 8.3 may fail.

One important consequence of Theorem 8 is that cooperative systems may have exponentially long cycles, which can be considered a form of chaotic behavior in discrete systems.

Corollary 9 *Let $p > 1$ and let c be an arbitrary real number with $1 < c < p$. Then for sufficiently large n , there exist n -dimensional cooperative p -discrete systems with periodic orbits of length $> c^n$.*

Proof. By Proposition 4, for each m there exist m -dimensional cooperative p -discrete systems with periodic orbits of length $d_{m,p}$. Now the conclusion follows from Lemma 6. ■

While Lemmas 5-7 are sufficient for deriving our conclusions about embeddings into cooperative systems, for completeness we will conclude this section with some sharper estimates of $d_{n,p}$. The following result is an application of a local central limit theorem.

Proposition 10 For arbitrary $p > 1$ and $\sigma^2 = \frac{1}{12}(p-1)(p+1)$:

$$\lim_{n \rightarrow \infty} d_{n,p} \left(\frac{p^n}{\sqrt{2\pi n \sigma^2}} \right)^{-1} = 1,$$

Proof. Let Y_1, \dots, Y_n be i.i.d. random variables, each of which can take any of the values $0, \dots, p-1$ with equal probability $1/p$. Then $E(Y_i) = (p-1)/2$, $\sigma^2(Y_i) = \frac{1}{12}(p-1)(p+1)$, for every i , and $d_{n,p} = p^n \cdot P(\sum_{i=1}^n Y_i = \lfloor n(p-1)/2 \rfloor)$.

Define $X_i := Y_i - (p-1)/2$, $S_n = \sum_{i=1}^n X_i$. We will use the notation of Section 2.5 in [4], in order to use Theorem 2.5.2 in that textbook. Set

$$x =: \left(\left\lfloor \frac{n(p-1)}{2} \right\rfloor - \frac{n(p-1)}{2} \right) \frac{1}{\sqrt{n}}, \quad p_n(x) := P(S_n/\sqrt{n} = x),$$

and note that $p_n(x) = P(\sum_{i=1}^n Y_i - n(p-1)/2 = \sqrt{n}x) = d_{n,p}p^{-n}$.

Let $\Phi(x) := (2\pi\sigma^2)^{-1/2}e^{-x^2/(2\sigma^2)}$. Then according to Theorem 2.5.2, $|\sqrt{n}p_n(x) - \Phi(x)| \rightarrow 0$ as $n \rightarrow \infty$. But evidently $\Phi(x) \rightarrow \Phi(0)$, since $x = x(n) \rightarrow 0$. Therefore $\sqrt{n}p_n(x) = \sqrt{n}d_{n,p}p^{-n} \rightarrow \Phi(0)$. This immediately implies the result. ■

Corollary 11 There exists constants $c_1, c_2 > 0$, independent of n and p , such that for arbitrary $p > 1$ there exists $n_0(p)$ such that for all $n \geq n_0(p)$:

$$c_1 \frac{p^{n-1}}{\sqrt{n}} \leq d_{n,p} \leq c_2 \frac{p^{n-1}}{\sqrt{n}}$$

Proof. Let $\sigma^2(p) = \frac{1}{12}(p-1)(p+1)$ be as in the above result. One can compute $\sigma(p)/p = \frac{1}{\sqrt{12}}\sqrt{1-p^{-2}}$. Therefore $\frac{1}{4}p \leq \sigma(p) \leq \frac{1}{\sqrt{12}}p$, for all $p > 1$. Let $\epsilon > 0$ be an arbitrarily small number. Then for all $n \geq n_0 = n_0(p)$ Proposition 10 implies:

$$d_{n,p} \leq (1 + \epsilon) \frac{p^n}{\sqrt{2\pi n \sigma(p)}} \leq \frac{1 + \epsilon}{\frac{1}{4}\sqrt{2\pi}} \frac{p^{n-1}}{\sqrt{n}}.$$

The second inequality is obtained analogously. ■

4 Smale extensions

Assume $\Pi = \prod_{i=1}^n L_i$ and each L_i has a smallest and a largest element. Let $A \subseteq L$ and let $\gamma : A \rightarrow L$. We say that γ is *cooperative* if for all $x, y \in A$ the implication $x \leq y \rightarrow \gamma(x) \leq \gamma(y)$ holds. Clearly, if A is unordered, then γ is cooperative, and since L has a largest and a smallest element, the construction used in the proof of Lemma 3 implies that γ can be extended to a cooperative function on Π . However, the construction used in this proof is too crude to allow for such extensions if A contains comparable elements. Here we use a different construction to show that any cooperative partial function on Π can be extended to a cooperative function on Π .

Lemma 12 *Let $\Pi = \prod_{i=1}^n L_i$, where $(L_i, <)$ is complete in the sense that every subset of L_i has a supremum and an infimum in L_i . Let $A \subseteq \Pi$, and let $\gamma : A \rightarrow \Pi$ be cooperative. Then there exists a cooperative $g : \Pi \rightarrow \Pi$ such that $\gamma = g \upharpoonright A$.*

Proof. Let Π, A, γ be in the assumption. First note that we may wlog assume that for all $z \in \Pi$ there exists $x \in A$ such that $x \leq z$ or $z \leq x$. If not, then extend A to a set A^* with this property and such that $A^* \setminus A$ is unordered and each $x \in A^* \setminus A$ is incomparable with each $z \in A$. Extend γ to $\gamma^* : A^* \rightarrow \Pi$ in an arbitrary way, and note that γ^* must still be cooperative.

Given $z \in \Pi$, define $U(z) := \{x \in A : x \geq z\}$ and $\Pi_U := \{z \in \Pi : U(z) \neq \emptyset\}$. Note that $A \subseteq \Pi_U$. Let $\Pi_L := \Pi \setminus \Pi_U$, and for all $z \in \Pi_L$ define $L(z) := \{x \in \Pi_U : x \leq z\}$. Note that our assumption on A implies that $L(z) \neq \emptyset$ for all $z \in \Pi_L$. Let $\gamma(U(z)) := \{\gamma(x) : x \in U(z)\}$.

Now define $g(z) := \inf \gamma(U(z))$ for $z \in \Pi_U$ and let $g(L(z)) := \{g(x) : x \in L(z)\}$ for $z \in \Pi_L$. Finally, define $g(z) := \sup g(L(z))$ for $z \in \Pi_L$.

Claim 13 *The map g defined above is cooperative and satisfies $g \upharpoonright A = \gamma$.*

Proof. By completeness of \leq on each L_i , infima and suprema under the cooperative order of nonempty subsets of Π exist and are elements of Π . Thus g is well defined.

Suppose that $z \in A \subseteq \Pi_U$. Then $\gamma(z) \leq \gamma(x)$ for every $x \in U(z)$ by cooperativity of γ , hence $\gamma(z) = \inf \gamma(U(z)) = g(z)$. Thus $g \upharpoonright A = \gamma$.

To see that g is cooperative, let $y, z \in \Pi$ be such that $y \leq z$. If $y, z \in \Pi_U$, then $U(z) \subseteq U(y)$, and hence $g(y) = \inf \gamma(U(y)) \leq \inf \gamma(U(z)) = g(z)$. If $y, z \in \Pi_L$, then $L(y) \subseteq L(z)$, and hence then $g(y) = \sup g(L(y)) \leq \sup g(L(z)) = g(z)$. The only other possibility consistent with $y \leq z$ is $z \in \Pi_L$ and $y \in \Pi_U$. In this case $y \in L(z)$, and hence $g(y) \leq \sup g(L(z)) = g(z)$. ■

For reasons that will become clear shortly, we will refer to the function g constructed in the proof of Lemma 12 as the *Smale extension* of γ .

Now suppose Π is either $\{0, \dots, p-1\}^n$ or $[0, 1]^n$ with the natural order, and let A be a hyperplane of the form $A = \{x \in \Pi : S(x) = r\}$. If A is nonempty (which will happen for suitable values of r), then A is a maximal incomparable subset of Π .

Note that if A is a hyperplane as above, then the definition of the Smale extension g of γ can be written as

$$g(z) = \begin{cases} \inf \gamma(U(z)), & \text{for } z \in \Pi_U, \\ \sup \gamma(L(z)), & \text{for } z \in \Pi_L. \end{cases} \quad (4)$$

For $x \in \Pi$, let $\|x\| = \max\{|x_1|, \dots, |x_n|\}$ be the sup-norm in \mathbb{R}^n .

Lemma 14 *Suppose Π is either $\{0, \dots, p-1\}^n$ or $[0, 1]^n$ with the natural order, and $A = \{x \in \Pi : S(x) = r\}$ is a nonempty hyperplane. Let $\gamma, \gamma_1 : A \rightarrow \Pi$ be cooperative, and let $\varepsilon, \delta > 0$ be such that*

$$\forall x, y \in A \quad \|x - y\| < (2n+1)\delta \Rightarrow \|\gamma(x) - \gamma_1(y)\| < \frac{\varepsilon}{3}. \quad (5)$$

Let g, g_1 be the Smale extensions of γ, γ_1 . Then

$$\forall x, y \in \Pi \quad \|x - y\| < \delta \Rightarrow \|g(x) - g_1(y)\| < \varepsilon. \quad (6)$$

Proof. Let $A, \gamma, \varepsilon, \delta$ be as in the assumption. First note that

$$\forall x, y \in \Pi \forall a \in U(x) \exists b \in U(y) \quad \|a - b\| \leq (n - 1)\|x - y\|. \quad (7)$$

To see this, let $x, y \in \Pi_U$ and $a \in A$ with $x \leq a$. Let $b \in U(y)$ be such that $\sum |b_i - a_i|$ is minimal. Such b exists by compactness of A . Note that we must have $b_i \leq \max\{a_i, y_i\}$ for all $i \in \{1, \dots, n\}$: If not, since $S(a) = S(b)$, there would be some j with $y_j \leq b_j < a_j$. Letting $\beta = \min\{b_i - \max\{a_i, y_i\}, a_j - b_j\}$ and $b_i^* = b_i - \beta$, $b_j^* = b_j + \beta$, and $b_k^* = b_k$ for all $k \neq i, j$, we would have $y \leq b^* \in A$ and $\sum |b_i^* - a_i| < \sum |b_i - a_i|$, contradicting the choice of b . Thus $|b_i - a_i| \leq \|x - y\|$ for all i with $b_i > a_i \geq x_i$. Now consider i with $y_i \leq b_i < a_i$. In this situation we must have $a_i - b_i \leq \sum \max\{0, b_j - a_j\} \leq (n - 1)\|x - y\|$. The inequality $\|a - b\| \leq (n - 1)\|x - y\|$ follows.

Furthermore, note that

$$\forall x \in \Pi_U \forall y \in \Pi_L \forall a \in U(x) \forall b \in L(y) \quad \|a - b\| \leq (2n + 1)\|x - y\|. \quad (8)$$

To see this, let $x \in \Pi_U, y \in \Pi_L, a \in U(x), b \in L(y)$. Fix $i \in \{1, \dots, n\}$. Then $a_i - x_i \leq S(a) - S(x) = r - S(x) \leq S(y) - S(x) \leq n\|y - x\|$. Similarly, $y_i - b_i \leq S(y) - S(b) = S(y) - r \leq S(y) - S(x)$. Now it follows from the triangle inequality that $|a_i - b_i| \leq 2(S(y) - S(x)) + |y_i - x_i| \leq (2n + 1)\|y - x\|$.

Now let x, y be such that $\|x - y\| < \delta$.

First assume that $x, y \in \Pi_U$. Fix $i \in n$, and let $a \in U(x)$ be such that $|(g(x))_i - (\gamma(a))_i| < \varepsilon/3$. Such a exists by (4). Choose $b \in U(y)$ as in (7). It follows from (5) that $\|\gamma(a) - \gamma_1(b)\| < \varepsilon/3$. In particular, $|(\gamma(a))_i - \gamma_1(b)_i| < \varepsilon/3$. Since $y \leq b$, definition (4) implies that $(g_1(y))_i \leq (\gamma_1(b))_i$, and the inequality $(g_1(y))_i < (g(x))_i + 2\varepsilon/3$ follows. By symmetry of the assumption, we also will have $(g(x))_i < (g_1(y))_i + 2\varepsilon/3$ in this case.

By the alternative definition (4) of the Smale embedding, the argument in the case when $x, y \in \Pi_L$ is dual.

Now assume $x \in \Pi_U$ and $y \in \Pi_L$. Fix $i \in n$, and let $a \in U(a)$ and $b \in L(b)$ be such that $|(g(x))_i - (\gamma(a))_i| < \varepsilon/3$ and $|(g_1(y))_i - (\gamma_1(b))_i| < \varepsilon/3$. By (8), $\|a - b\| \leq (2n + 1)\|y - x\| < (2n + 1)\delta$, and (5) implies that $|(\gamma(a))_i - \gamma_1(b)_i| < \varepsilon$. Now (6) follows from the triangle inequality.

The argument in the case when $x \in \Pi_L$ and $y \in \Pi_U$ is symmetric. ■

By letting $\gamma = \gamma_1$ in Lemma 14, we immediately get the following:

Corollary 15 *Suppose $\Pi = [0, 1]^n$ with the natural cooperative order, and $A = \{x \in \Pi : S(x) = r\}$ for some $0 \leq r \leq 1$. Let $\gamma : A \rightarrow \Pi$ be cooperative, and let $g : \Pi \rightarrow \Pi$ be the Smale extension of γ .*

(i) *If γ is continuous, so is g .*

(ii) *If γ is Lipschitz-continuous with Lipschitz constant ℓ , then g is Lipschitz continuous with Lipschitz constant $\leq (6n + 3)\ell$.*

Now consider any discrete-time dynamical system $([0, 1]^n, f)$, let $A = \{x \in [0, 1]^{n+1} : S(x) = (n+1)/2\}$, and let $\phi : [0, 1]^n \rightarrow A$ be a Lipschitz-continuous homeomorphism. Let $\gamma : A \rightarrow A$ be such that $\gamma(\phi(x)) = \phi(f(x))$ for all $x \in [0, 1]^n$. If f is (Lipshitz)-continuous, then so is γ , and Corollary 15 implies that ϕ is a (Lipshitz)-continuous embedding of $([0, 1]^n, f)$ into a discrete-time dynamical system $([0, 1]^{n+1}, g)$ for which g is (Lipshitz)-continuous. This is analogous to Smale's famous embedding theorem for C^1 -systems [19] and is our motivation for calling the function g of Lemma 12 the *Smale extension* of γ .

If $([0, 1]^n, f)$ and $(\{0, \dots, p-1\}^n, f_1)$ are two discrete-time systems and $\varepsilon > 0$, then we will say that f_1 is an ε -approximation of f if $\|\frac{1}{p-1}f_1(\lfloor (p-1)x \rfloor) - f(x)\| < \varepsilon$ for all $x \in [0, 1]^n$.

Let A be as in the previous paragraph, let $D = \{y \in \{0, \dots, p-1\}^{n+1} : S(y) = \lfloor (n+1)(p-1)/2 \rfloor\}$, and define $D^* := \{a \in A : (p-1)a \in D\}$. It is clear that if f is continuous, $\delta > 0$ is given, $\beta > 0$ is sufficiently small relative to δ , p is odd and sufficiently large, and if ϕ, γ are as in the previous paragraph, then there exist:

- a β -approximation $(\{0, \dots, p-1\}^n, f_1)$ of $([0, 1]^n, f)$,
- and a function $\gamma_1 : A \rightarrow D^*$ such that $\|\gamma(y) - \gamma_1(y)\| < \delta$ for all $y \in A$,
- a function $\phi_1 : \{0, \dots, p-1\}^n \rightarrow D^*$ such that $\|\phi(x/(p-1)) - \phi_1(x)\| < \delta$ and $\gamma_1(\phi_1(x)) = \phi_1(f(x))$ for all $x \in \{0, \dots, p-1\}^n$.

Now let $\varepsilon > 0$ be given. By Lemma 14, if we choose the above objects for δ sufficiently small relative to ε and if g is the Smale extension of γ , while g_1 is the Smale extension of γ_1 , then $\|g(y) - g_1(y)\| < \varepsilon$ for all $y \in [0, 1]^{n+1}$. Let $g_1^*(x) := (p-1)g_1(x/(p-1))$ for all $x \in \{0, \dots, p-1\}^{n+1}$. From the definition of the Smale extension it follows that g_1^* maps $x \in \{0, \dots, p-1\}^{n+1}$ into itself, and the inequality $\|g(y) - g_1(y)\| < \varepsilon$ implies that g_1^* is an ε -approximation of g . Moreover, we will have $g_1^*(\phi_1(x)) = \phi_1(f_1(x))$ for all $x \in \{0, \dots, p-1\}^n$. However, we cannot necessarily assume that ϕ_1 is a cooperative embedding of $(\{0, \dots, p-1\}^n, f_1)$ into $(\{0, \dots, p-1\}^{n+1}, g_1^*)$, since the results of Section 3 indicate that the function ϕ_1 may not be injective.

5 Almost cooperative systems

Cooperative systems are so named because increasing the value of one variable tends to increase the values of other variables in the system. For instance, in the continuous case a condition equivalent to the cooperativity of the system (2) is $\partial f_i / \partial x_j(x) \geq 0$ for $i \neq j$ [20]. It has been conjectured that a system might have amenable properties if it is 'almost cooperative,' i.e. if the latter condition is satisfied with the exception of a single pair $i \neq j$ (see the concept of *consistency deficit* in [23]).

We can define a discrete counterpart of this notion as follows. Let $x \in \Pi = \prod_{i=1}^n \{0, \dots, p_i-1\}^n$, and let $i \in \{1, \dots, n\}$. Define $x^{i+} \in \Pi$ by letting $(x^{i+})_i = \min\{x_i + 1, p_i - 1\}$ and $(x^{i+})_j = x_j$ for $j \neq i$. Similarly, define $x^{i-} \in \Pi$ by letting $(x^{i-})_i = \max\{x_i - 1, 0\}$ and $(x^{i-})_j = x_j$ for $j \neq i$. It is easy to see that cooperativity of a system (Π, g) is equivalent to the condition that

$$\forall x \in \Pi \quad (g(x^{i-}))_j \leq (g(x))_j \leq (g(x^{i+}))_j \quad (9)$$

for all $i, j \in \{1, \dots, n\}$. Let us call (Π, g) *almost cooperative* if condition (9) holds with the exception of exactly one pair $\langle i^*, j^* \rangle$ with $i^* \neq j^*$ for which an order-reversal takes place:

$$\forall x \in \Pi \quad (g(x^{i^*-}))_{j^*} \geq (g(x))_{j^*} \geq (g(x^{i^*+}))_{j^*}. \quad (10)$$

One might expect that almost cooperative p -discrete systems are similar to cooperative systems. In particular, one might expect that m -dimensional almost cooperative Boolean systems can always be embedded into cooperative Boolean systems of dimension $m + 1$. However, this is not the case.

Consider the following simple example with $n = 2$. In this case $|\Pi| = 4$, and $d_{3,2} = \binom{3}{2} = 3$. Define $g(x_1, x_2) := (1 - x_2, x_1)$, so that $g(0, 0) = (1, 0)$, $g(1, 0) = (1, 1)$, $g(1, 1) = (0, 1)$, and $g(0, 1) = (0, 0)$. The system (Π, g) consists of a single orbit of length 4. By Proposition 4, this system cannot be embedded into any cooperative Boolean system of dimension 3. Moreover, note that x_1 promotes the increase of the variable x_2 , while x_2 inhibits the variable x_1 . Thus condition (9) holds with the exception of the pair $\langle i, j \rangle = \langle 2, 1 \rangle$ and hence the system is almost cooperative.

6 Strong Cooperativity for Finite Discrete Systems

Throughout the remaining two sections we will assume that the state space Π of our dynamical system is finite. Wlog this means that $\Pi = \prod_{i=1}^n \{0, \dots, p_i - 1\}$, where the p_i 's are integers such that $p_1 \geq \dots \geq p_i \geq \dots \geq p_n > 1$. Of course, p -discrete systems are exactly those among the above systems for which $p_i = p > 1$ for all i .

Our first task is to come up with a suitable counterpart of strong cooperativity for such systems. Let us write $x < y$ if $x \leq y$ in the cooperative order but $x \neq y$, and let us write $x \ll y$ if $x_i < y_i$ for all $i = 1, \dots, n$. Recall that a continuous system is strongly cooperative if for every two initial conditions $x(0) < y(0)$ we have $x(t) \ll y(t)$ for all $t > 0$. Perhaps the most straightforward adaptation of this definition to finite discrete systems would be the following:

$$\forall x(0) < y(0) \exists t_0 > 0 \forall t \geq t_0 \quad x(t) \ll y(t). \quad (11)$$

This property is commonly known as eventual strong cooperativity [20]. Unfortunately, no finite discrete system of dimension $n > 1$ satisfies (11). To see this, note that if $n > 1$, then there exist $x^0(0) < x^1(0) < \dots < x^{p_1}(0)$. On the other hand, $x(t) \ll y(t)$ implies $n + S(x(t)) \leq S(y(t))$. Now (11) would imply $S(x^{p_1}(t)) \geq np_1$ for sufficiently large t . But this is a contradiction, since $S(x) \leq (p_1 - 1)n$ for all $x \in \Pi$.

So let us define a weaker discrete version of strong cooperativity. Consider the following properties of a finite cooperative system.

$$\forall x(0) < y(0) \quad S(y(0) - x(0)) \leq S(y(1) - x(1)). \quad (12)$$

$$\forall x(0) < y(0) \quad x(1) < y(1). \quad (13)$$

$$\forall x(0) \quad S(x(1)) = S(x(0)). \quad (14)$$

Lemma 16 *For any finite cooperative discrete system conditions (12), (13) and (14) are equivalent.*

Proof. Clearly, condition (12) implies condition (13) and condition (14) implies condition (12) for cooperative systems.

Now assume an n -dimensional finite system (Π, g) satisfies condition (13). Let $x(0) \in \Pi$, let $S := \sum_{i=1}^n (p_i - 1)$, and consider initial states $x^0(0) < x^1(0) < \dots < x^S(0)$ such that $x(0) = x^{S(x)}(0)$. By (13), $x^0(1) < x^1(1) < \dots < x^S(1)$. It follows that $S(x^i(1)) = S(x^i(0))$ for all i , and in particular (14) holds. ■

We will say that a finite cooperative system (Π, g) is *strongly cooperative* if it satisfies conditions (12), (13) and (14).

For example, let $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ be a permutation, and define g_π by $(g_\pi(x))_{\pi(i)} = x_i$ for all $x \in \Pi$ and $i \in \{1, \dots, n\}$. Note that if $\Pi = \{0, \dots, p-1\}^n$, then g_π maps Π into Π for all permutations π of $\{1, \dots, n\}$, but in general this will be the case only for some but not for all permutations. If g_π does map Π into Π , then (Π, g_π) is a strongly cooperative system.

The *order* of a permutation π is the smallest integer $r > 0$ such that π^r is the identity. Let $R(n)$ be the maximum order of a permutation π of $\{1, \dots, n\}$. It can be shown that $R(n) = e^{\sqrt{n \ln n}(1+o(1))}$ as $n \rightarrow \infty$ [12]. In particular, note that $R(n)$ grows *subexponentially* in n , that is, for every $b > 1$ and sufficiently large n we will have $R(n) < b^n$.

Theorem 17 *Suppose $(\prod_{i=1}^n \{1, \dots, p_i - 1\}^n, g)$ is an n -dimensional strongly cooperative finite discrete system, and let $N = \sum_{i=1}^n (p_i - 1)$. Then each periodic orbit in (Π, g) has length at most $R(N)$.*

Proof. Note that for any permutation π , the length of any periodic orbit of (Π, g_π) cannot exceed the order of π . However, not all strongly cooperative finite systems are of the form (Π, g_π) for some permutation π . For example, if $n = 2$ and $g(\emptyset) = \emptyset$, $g(\{1\}) = g(\{2\}) = \{1\}$ and $g(\{1, 2\}) = \{1, 2\}$, then g is a strongly cooperative Boolean system, but not of the form g_π for any permutation π .

Fortunately, Lemma 18 below suffices for the proof of our theorem in the Boolean case when $p_i = 2$ for all i and hence $N = n$. A state in an attractor of a dynamical system, i.e., a state that is not transient, will be called *persistent*.

Lemma 18 *Let (Π, g) be a strongly cooperative n -dimensional Boolean system. Then there exists a permutation π of $\{1, \dots, n\}$ such that $g(x) = g_\pi(x)$ for each persistent state x of (Π, g) .*

Proof. We will prove the lemma by induction over n . Note that it is trivially true for $n = 1$. Now fix $n > 1$, assume the lemma is true for all $k < n$ and let (Π, g) be as in the assumption. We will identify elements x of Π with subsets of the set $\{1, \dots, n\}$ and write $|x|$ instead of $S(x)$. Note that g maps one-element subsets of $\{1, \dots, n\}$ to one-element subsets. More

precisely, there exists a function $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that $g(\{i\}) = \{\sigma(i)\}$ for all i . In general, σ does not need to be a bijection. However, if I is the set of all i such that $\{i\}$ is a persistent state of our system, then $I \neq \emptyset$ and $\sigma \upharpoonright I$ is a permutation of I . Now strong cooperativity of g implies that $g(x) = g_{\sigma \upharpoonright I}(x)$ for all $x \subseteq I$. Thus if $I = \{1, \dots, n\}$, we are done. If not, then define for $y \in \Pi$ such that $I \cap y = \emptyset$:

$$f(y) = g(y \cup I) \setminus I.$$

Since $g(I) = g_{\sigma \upharpoonright I}(I) = I$, the function f is strongly cooperative on the set of all subsets of $J := \{1, \dots, n\} \setminus I$. By the inductive assumption, there exists a permutation ϱ of J such that $f(y) = g_{\varrho}(y)$ for all persistent states in the system defined by f . Note that $\pi := (\sigma \upharpoonright I) \cup \varrho$ is a permutation of $\{1, \dots, n\}$.

Now consider any $x = x(0) \in \Pi$. By strong cooperativity we have

$$|x| = |g(x)| = |g(x) \cap I| + |g(x) \cap J|.$$

On the other hand, $|x \cap I| = |g(x \cap I)| \leq |g(x) \cap I|$ because $g(x \cap I) \subseteq g(I) = I$. It follows that $|x(t) \cap J|$ is nonincreasing along the trajectory of $x(0)$. In particular, for every persistent state x we must have $g(x \cap I) = g(x) \cap I$ and hence $|x \cap J| = |g(x) \cap J|$.

It must also be the case that $g(x) \cap J \subseteq f(x \cap J)$. Since $|f(x \cap J)| = |x \cap J|$ by strong cooperativity of f , we must have $g(x) \cap J = f(x \cap J)$ for every persistent state x of (Π, g) . It follows that if x is a persistent state of (Π, g) , then $x \cap J$ is a persistent state of (Σ, f) . Thus $g(x) = g_{\sigma \upharpoonright I}(x \cap I) \cup g_{\varrho}(x \cap J) = g_{\pi}(x)$. \blacksquare

Now consider the general case where $p_i \geq 2$ for all i . Unfortunately, we cannot hope to prove the exact analogue of Lemma 18. To see this, consider the system $(\{0, 1, 2\}^2, g)$, where $g(0, 0) = [0, 0]$, $g(0, 1) = [1, 0]$, $g(1, 0) = [0, 1]$, $g(1, 1) = g(2, 0) = (0, 2) = [1, 1]$, $g(1, 2) = [1, 2]$, $g(2, 1) = [2, 1]$, $g(2, 2) = [2, 2]$. This system is clearly strongly cooperative. If π were a permutation as in Lemma 18, then we would need $\pi(0) = 1$ and $\pi(1) = 0$ because both $[0, 1]$ and $[1, 0]$ are persistent states. On the other hand, $[2, 1]$ and $[1, 2]$ are persistent steady states, so this would force π to be the identity.

Now let $\Pi = \prod_{i=1}^n \{1, \dots, p_i - 1\}^n$ and let (Π, g) , N be as in the assumption of Theorem 17. Let $\Sigma := \{0, 1\}^N$. For each $i \in \{1, \dots, n\}$ let J_i be the set of integers j such that $\sum_{k=1}^{i-1} (p_k - 1) < j \leq \sum_{k=1}^i (p_k - 1)$. For $1 \leq \ell \leq |J_i| = p_i - 1$ let $j(i, \ell)$ be the ℓ -th element of J_i . Define a map $\psi : \Pi \rightarrow \Sigma$ so that for $i \in \{1, \dots, n\}$ and $\ell \in \{1, \dots, p_i - 1\}$ we have $\psi(x)_{j(i, \ell)} = 1$ iff $x_i \geq \ell$. Clearly, ψ is an injection. For $y \in \Sigma$ define $z(y)$ by $z(y)_{j(i, \ell)} = 1$ iff $\ell \leq |\{\ell' : y_{j(i, \ell')} = 1\}|$. Note that $z(y)$ is always in the range of ψ , and $z(y) = y$ whenever y is already in the range of ψ . Moreover, the function $y \mapsto z(y)$ is strongly cooperative. Now define $f : \Sigma \rightarrow \Sigma$ so that $f(y) = f(z(y))$ for all y and $f(\psi(x)) = \psi(g(x))$ for all $x \in \Pi$.

Then ψ is an embedding of (Π, g) into (Σ, f) . Moreover, if (Π, g) is strongly cooperative, then so is (Σ, f) . Since the lemma is true for $p = 2$, each periodic orbit in (Σ, f) has length at most $R(N)$, and since ψ is an embedding, the same must be true for (Π, g) . \blacksquare

Question 19 Suppose (Π, g) is an arbitrary n -dimensional strongly cooperative finite discrete system. Can the system have a periodic orbit of length greater than $R(n)$? What if we assume in addition that (Π, g) is p -discrete for some $p > 2$?

Our results can perhaps be considered analogues of the result in [15] for discrete-time continuous-space strongly cooperative systems. Our Theorem 17 gives a nontrivial, subexponential bound on the lengths of periodic orbits of strongly cooperative finite discrete systems. Moreover, strong cooperativity implies that ordered orbits in finite discrete systems are fairly robust, as shown in the next result.

Lemma 20 *Consider a strongly cooperative finite discrete system (1), and let $x(0)$ and $y(0)$ be two arbitrary initial conditions (i.e. not necessarily ordered). Then $S(|y(t) - x(t)|) \leq S(|y(0) - x(0)|)$ for all $t > 0$.*

Proof. Suppose first that $S(|y(0) - x(0)|) = 1$. Then necessarily the two initial conditions are ordered; suppose without loss of generality $x(0) < y(0)$. By condition (14) and cooperativity, $S(|y(0) - x(0)|) = S(y(0) - x(0)) = S(y(0)) - S(x(0)) = S(y(1)) - S(x(1)) = S(|y(1) - x(1)|)$.

If $S(|y(0) - x(0)|) = k > 1$, then there exists a sequence of states $x = x^0, x^1, \dots, x^k = y$, such that $S(|x^{j+1} - x^j|) = 1$ for every j . Then $S(|y(1) - x(1)|) \leq S(|x^k(1) - x^{k-1}(1)|) + \dots + S(|x^1(1) - x^0(1)|) = S(|x^k(0) - x^{k-1}(0)|) + \dots + S(|x^1(0) - x^0(0)|) = k = S(|y(0) - x(0)|)$. \blacksquare

In other words, small perturbations of initial conditions don't amplify along the trajectory, which implies an analogue of Lyapunov stability for all attractors.

7 Cooperative Irreducible Systems and Long Periodic Orbits

In this section we will explore several possible discrete counterparts of the notion of irreducible cooperative C^1 -systems and will show how these conditions relate to strong cooperativity and what bounds they impose on the lengths of periodic orbits.

Recall that a digraph (directed graph) $G = (V, A)$ is *strongly connected* if every node w in V can be reached via a directed path from every node $v \in V$.

Now let us define discrete analogues of irreducible cooperative systems by associating directed graphs $G = (\{1, \dots, n\}, A)$ with a cooperative system (Π, g) . Recall that in the definition of irreducible cooperative C^1 -systems, an arc $\langle i, j \rangle$ was included in the arc set A iff $Df(x)_{ij} > 0$ on \mathbb{R}^n , where $Df(x)$ is the Jacobian of $f(x)$, and the system was called irreducible if the resulting directed graph G on \mathbb{R}^n was strongly connected. Alternatively, a digraph G_x can be defined locally for every $x \in \mathbb{R}^n$ by letting $\langle i, j \rangle$ be an arc in G_x if and only if $Df(x)_{ij} > 0$. A cooperative C^1 -system in which G_x is strongly connected for every $x \in \mathbb{R}^n$ is still strongly cooperative; see for instance Corollary 3.11 in [9].

Recall the definitions of x^{i-} and x^{i+} from Section 5. For an n -dimensional finite discrete system (Π, g) and $x \in \Pi$, let us define a directed graph $G_x^* = (\{1, \dots, n\}, A_x^*)$ by including an arc $\langle i, j \rangle \in A_x^*$ iff $g(x)_j < g(x^{i+})_j$ or $g(x^{i-})_j < g(x)_j$. Moreover, let us define a directed graph $G_x = (\{1, \dots, n\}, A_x)$ by including an arc $\langle i, j \rangle \in A_x$ iff $\langle i, j \rangle \in A_x^*$ and if $0 < x_i < p_i - 1$, then $g(x^{i-})_j < g(x)_j < g(x^{i+})_j$.

Let us call the system (Π, g) *strongly irreducible* if $(\{1, \dots, n\}, \bigcap_{x \in \Pi} A_x)$ is strongly connected, *strongly semi-irreducible* if $(\{1, \dots, n\}, \bigcap_{x \in \Pi} A_x^*)$ is strongly connected, *irreducible*

if G_x is strongly connected for all $x \in \Pi$, and *weakly irreducible* if $(\{1, \dots, n\}, \bigcup_{x \in \Pi} A_x^*)$ is strongly connected.

Note that for Boolean systems, $A_x^* = A_x$ for all states x ; hence the notions of strong irreducibility and strong semi-irreducibility coincide for Boolean systems. While both strong irreducibility and strong semi-irreducibility are plausible counterparts of irreducibility in cooperative C^1 -systems, we will see that these two notions have dramatically different implications for the dynamics of non-Boolean finite discrete systems.

In analogy to continuous systems, one would expect that irreducibility of cooperative discrete systems would imply strong cooperativity and would put nontrivial bounds on the length of periodic orbits, but weak irreducibility would not, since this condition can be guaranteed in local neighborhoods of x 's that are far away from the attractor. Therefore we will also consider the following properties. Let (Π, g) be an n -dimensional finite discrete system, and let X be an attractor. We say that the system (Π, g) is *strongly irreducible along X* if $(\{1, \dots, n\}, \bigcap_{x \in X} A_x)$ is strongly connected, *irreducible along X* if G_x is strongly connected for all $x \in X$, and *weakly irreducible along X* if $(\{1, \dots, n\}, \bigcup_{x \in X} A_x^*)$ is strongly connected. Note that (strongly) irreducible systems are (strongly) irreducible along every attractor. In contrast, a system that is weakly irreducible along at least one attractor is already weakly irreducible.

Let π be a permutation of $\{1, \dots, n\}$. Recall the definition of the function g_π from the previous section. Note that a finite discrete system (Π, g_π) is strongly irreducible iff (Π, g_π) is weakly irreducible along some attractor iff the permutation π is cyclic.

Theorem 21 *Suppose (Π, g) is a cooperative irreducible finite discrete system. Then (Π, g) is strongly cooperative and strongly irreducible. Moreover, there exists a cyclic permutation π of $\{1, \dots, n\}$ such that $g = g_\pi$.*

Proof. Let $x, y \in \Pi$ be such that $x < y$. Pick $i \in \{1, \dots, n\}$ such that $x < x^{i+} \leq y$. Then there exists some j with $\langle i, j \rangle \in A_x$; otherwise G_x could not be strongly connected. Thus $g(x)_j < g(x^{i+})_j \leq g(y)$ by cooperativity, and condition (13) follows. Thus (Π, g) is strongly cooperative.

Now let us consider $A_{\vec{0}}$. Let us write $\{i\}$ for the $x \in \Pi$ with $x_i = 1 = S(x)$ and call such x a *singleton*. Note that $\langle i, j \rangle \in A_{\vec{0}}$ iff $g(\{i\})_j > 0$. By strong cooperativity, $g(\{i\})$ is a singleton, and it follows that the outdegree of each i in $G_{\vec{0}}$ is at most one. Strong connectedness of $G_{\vec{0}}$ now implies that the in- and outdegrees in $G_{\vec{0}}$ of all nodes are exactly one. Let $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ be defined by $\pi(i) = j$ iff $\langle i, j \rangle \in A_{\vec{0}}$. Then π is a permutation. Moreover, if π could be decomposed into nonempty pairwise disjoint cycles, then $G_{\vec{0}}$ would not be strongly connected. Thus π must be cyclic.

It remains to show that $g = g_\pi$. We will show this by induction over $S(x)$. If $S(x) = 0$, then $g(x) = x$ by strong cooperativity, hence $g(x) = g_\pi(x)$. By the way we defined π we also have $g(x) = g_\pi(x)$ whenever $S(x) = 1$.

Now let us assume $g(x) = g_\pi(x)$ for all x with $S(x) = k$, and let y be such that $S(y) = k + 1$. Then $y = x^{i+}$ for some i and x with $S(x) = k$. By the inductive assumption, $g(x) = g_\pi(x)$. If $x_i = 0$, then $(g_\pi(x))_{\pi(i)} = 0$ but we must have both $g(x) < g(y)$ and $g(\{i\}) = \{j\} \leq g(y)$, so $g_\pi(y) \leq g(y)$, and strong cooperativity implies $g(y) = g_\pi(y)$. If $x_i > 0$, then the definition of A_x implies that there must be j with $g(x^{i-})_j < g(x)_j <$

$g(x^{i+})_j$. But by inductive assumption, the only j with $g(x^{i-})_j < g(x)_j$ is $\pi(i)$, so we must also have $g(x)_j < g(x^{i+})_j$. It again follows that $g_\pi(y) \leq g(y)$, and hence $g(y) = g_\pi(y)$ by strong cooperativity. ■

Corollary 22 *Periodic orbits in cooperative irreducible n -dimensional finite discrete systems can have length at most n .*

Proof. The maximal order of a cyclic permutation on $\{1, \dots, n\}$ is n . ■

Corollary 22 gives a stronger bound than Theorem 17 does for strongly cooperative p -discrete systems. For Boolean systems we can prove the same bound under weaker assumptions.

Lemma 23 *Suppose (Π, g) is a strongly cooperative n -dimensional Boolean system, and let X be an attractor such that (Π, g) is weakly irreducible along X . Then $|X| \leq n$.*

Proof. Let (Π, g) and X be as in the assumptions. We will identify $x \in \Pi$ with subsets of $\{1, \dots, n\}$. By Lemma 18, there exists a permutation π of $\{1, \dots, n\}$, where n is the dimension of the system, such that $g(x) = g_\pi(x)$ for all $x \in X$. It suffices to show that π is cyclic. Let $I \subset \{1, \dots, n\}$ be the elements of a cycle of π such that $x \cap I \neq \emptyset$ for all $x \in X$ or $X = \{\emptyset\}$. Now suppose $I \neq \{1, \dots, n\}$, and let $J = \{1, \dots, n\} \setminus I$. Note that under these assumptions, for all $x \in X$ we have $|g(x \cap I)| = |g(x) \cap I|$ and $|g(x \cap J)| = |g(x) \cap J|$. Moreover, we must have $g(I) = I$.

We will reach a contradiction with weak irreducibility along X by showing that if $\langle i, j \rangle \in A_x$ for some $x \in X$, then we cannot have $i \in I$ and $j \in J$. Suppose that $\langle i, j \rangle \in A_x$ for $x \in X$ with $i \in I$ and $j \in J$. Then there exists y such that either $x < x \cup \{i\} = y$, $j \in g(y) \setminus g(x)$ or $y = x \setminus \{i\} < x$, $j \in g(x) \setminus g(y)$. Wlog assume the former; the proof in the latter case is analogous. Let $z = I \cup (x \cap J)$. Then $g(z) \supseteq g(I) = I$, and $g(z) \supseteq g(y) \supseteq g(x) \cup \{j\}$. It follows that $|g(z)| \geq |I| + |g(x) \cap J| + 1$, since $j \notin g(x)$. But this implies $|g(z)| > |z|$, contradicting the assumption of strong cooperativity. ■

For non-Boolean systems, the assumption of irreducibility in Theorem 21 or Corollary 22 cannot be replaced by the assumption of strong semi-irreducibility.

Example 24 *For every n there exists a cooperative strongly semi-irreducible 4-discrete system (Π, g) of dimension n that contains a periodic orbit of length $d_{n,2}$.*

Proof. Fix n , let $(\{0, 1\}^n, f)$ be a cooperative Boolean system with a periodic orbit of length $d_{n,2}$, and let π be a cyclic permutation of $\{1, \dots, n\}$. Let $\Pi = \{0, 1, 2, 3\}^n$ and define a function $g : \Pi \rightarrow \Pi$ as follows. Let $S = \{x \in \Pi : \min x = 0 \leq \max x < 3\}$, $M = \{x \in \Pi : 1 \leq \min x \leq \max x \leq 2\}$, and $L = \{x \in \Pi : \max x = 3\}$. For $x \in S$, let $g(x)_{\pi(i)} = 0$ whenever $x_i = 0$ and $g(x)_{\pi(i)} = 1$ whenever $x_i > 0$. For $x \in M$, let $g(x)_j = 1 + f(x - 1)$, and for $x \in L$ let $g(x)_{\pi(i)} = 2$ whenever $0 < x_i < 3$, $g(x)_{\pi(i)} = 0$ whenever $x_i = 0$, and $g(x)_{\pi(i)} = 3$ whenever $x_i = 3$.

Note that the restriction $g \upharpoonright M$ is isomorphic to f , hence the restriction of our system to M is cooperative and has a periodic orbit of length $d_{n,2}$. It also follows immediately

from the definitions that the restriction of our system to S as well as its restriction to L are cooperative. Moreover, consider $x \in S$, $y \in M$ and $z \in L$. Then $g(x) \leq g(y)$, and $x \leq z$ implies $g(x) \leq g(z)$. Similarly, $y \leq z$ implies $g(y) \leq g(z)$. Since no element of S can sit above an element of M or L , and no element of M can sit above an element of L , strong cooperativity of the whole system follows.

It remains to show that our system is strongly semi-irreducible. It suffices to show that if $\pi(i) = j$, then $\langle i, j \rangle \in A_x^*$ for all $x \in \Pi$. Fix x and i, j with $j = \pi(i)$. If $x_i = 3$, then $x \in L$ and $(g(x^{i-}))_j \leq 2 < 3 = (g(x))_j$. If $x_i = 2$, then $x^{i+} \in L$ and $(g(x))_j \leq 2 < 3 = (g(x^{i+}))_j$. Similarly, if $x_i = 1$, then $x^{i-} \in S \cup L$ and $(g(x^{i-}))_j = 0 < 1 \leq (g(x))_j$. Finally, if $x_i = 0$, then $x \in S \cup L$ and $(g(x))_j = 0 < 1 \leq (g(x^{i+}))_j$. ■

It turns out that for Boolean cooperative systems, strong irreducibility along an attractor X all by itself (without the assumption of strong cooperativity) puts tight bounds on the length of this attractor.

Theorem 25 *Suppose (Π, g) is an n -dimensional cooperative Boolean system, and let X be an attractor such that (Π, g) is strongly irreducible along X . Then $|X| \leq n$.*

Proof. Let (Π, g) and X be as in the assumption, and let $A = \bigcap_{x \in X} A_x$. We will identify $x \in \Pi$ with subsets of $\{1, \dots, n\}$. Note that if $\langle i, j \rangle \in A$, then for all $x \in X$ we have $i \in x$ iff $j \in g(x)$. By induction, if j can be reached by a directed path in G of length r , then $i \in x$ iff $j \in g^r(x)$. For each $1 \leq j \leq n$ define $\ell(j)$ as the length of the shortest directed path in G from 1 to j . Strong connectedness implies that $\ell(j)$ is well-defined and $1 \leq \ell(j) \leq n$ for all j . Now let $x(0) \in X$ and note that $x(r\ell(1) + s)_1 = (g^{r\ell(1)+s}(x))_1 = (g^s(x))_1 = (x(s))_1$ for all nonnegative integers r, s . More generally, $x(r\ell(1) + s + \ell(j))_j = x(s)_1$ for all r, s and j . Thus $(x(r\ell(1)))_j = (x(-\ell(j)))_j = (x(0))_j$ for all j , and hence $x(r\ell(1)) = x(0)$. The theorem follows. ■

Theorem 25 does not generalize to arbitrary cooperative p -discrete systems. In general, a p -discrete system may be strongly irreducible even along an attractor of exponential length.

Example 26 *For every n there exists a cooperative 6-discrete system (Π, g) of dimension n that is strongly irreducible along a periodic orbit of length $d_{n,2}$.*

Proof. The construction is similar to Example 24. Fix n and let $(\{0, 1\}^n, f)$ be a cooperative Boolean system where $D = \{x \in \{0, 1\}^n : S(x) = \lfloor n/2 \rfloor\}$ is a periodic orbit, and let π be a cyclic permutation of $\{1, \dots, n\}$. Let $\Pi = \{0, \dots, 5\}^n$ and let $M = \{1 + 3x : x \in D\}$. Define $h(1 + 3x) = 1 + 3f(x)$ for all $x \in D$. For $y \in M$ and $i \in \{1, \dots, n\}$, define $h(y^{i+}) = h(y)^{\pi(i)+}$ and $h(y^{i-}) = h(y)^{\pi(i)-}$. Let A be the set of $y \in \Pi$ for which we have defined h . Note that for $x, y \in M$ and $i, j \in \{1, \dots, n\}$ we can have $x^{i-} \leq y^{j+}$ only if $x = y$. Thus h is cooperative on A . By Lemma 12, there exists a cooperative function $g : \Pi \rightarrow \Pi$ so that $g \upharpoonright A = h$. Clearly, M is a periodic orbit of length $d_{n,2}$ of g , and it follows from our definitions that $\langle i, \pi(i) \rangle \in A_y$ for all $y \in M$. Thus (Π, g) is strongly irreducible along M . ■

The following example shows that assumption in Lemma 25 cannot be weakened to irreducibility along the attractor.

Example 27 For every $n \geq 1$ there exists an n -dimensional cooperative Boolean system (Π, g) and a periodic orbit X of length $d_{n,2}$ such that the system is irreducible along X .

Proof. The statement is trivially true for $n = 1$. So let $n > 1$, let D be as in (3) and let $f : D \rightarrow D$ be such that the system (D, f) is cyclic. Embed (D, f) into a cooperative Boolean system (Π, g) as in Lemma 3. Consider $x \in D$. Then $\langle i, j \rangle \in A_x$ iff either $i \in x$ and $j \in f(x)$ or $i \notin x$ and $j \notin f(x)$. Let $i \in \{1, \dots, n\}$. We will show that every node in $\{1, \dots, n\}$ can be reached from i by a directed path in G_x which implies strong connectedness of G_x . Assume $i \in x$; the proof in the case $i \notin x$ is dual. Then $\langle i, j \rangle \in A_x$ for all $j \in f(x)$. Now note that since $f(x) \neq x$ and both $x, f(x) \in D$, there must exist $j \in f(x) \setminus x$. Then $\langle j, k \rangle \in A_x$ for every $k \notin f(x)$, and it follows that every node in $\{1, \dots, n\}$ can be reached from i by a directed path of length at most two. ■

It is not in general true for every periodic attractor of length $d_{n,2}$ in an n -dimensional cooperative Boolean system that the system is irreducible along X . We have the following example.

Example 28 Fix $p > 1$, let n sufficiently large and let D be as in (3). Then there exists a function $\gamma : D \rightarrow D$ such that whenever (Π, g) is an n -dimensional cooperative p -discrete system with $g \upharpoonright D = \gamma$, then D is a periodic attractor of length $d_{n,p}$ such that $\bigcup_{x \in D} A_x^*$ contains all arcs $\langle i, j \rangle$ with $i \neq j$. In particular, the system (Π, g) is weakly irreducible along D . However, if g is the Smale extension of f , then (Π, g) is not irreducible along D .

Proof. Let $\langle n \rangle^2$ denote the set of all pairs $s = \langle i, j \rangle$, for arbitrary $1 \leq i < j \leq n$. For any $s = \langle i, j \rangle \in \langle n \rangle^2$, let $\pi_s : D \rightarrow D$ be the result of swapping the values of the components i and j . That is, for $d \in D$, let $(\pi_s d)_i := d_j$, $(\pi_s d)_j := d_i$, and $(\pi_s d)_k := d_k$ for all $k \neq i, j$.

Consider a function $a : \langle n \rangle^2 \rightarrow D$, and define $b : \langle n \rangle^2 \rightarrow D$ by $b(s) := \pi_s a(s)$. Similarly, consider another function $a' : \langle n \rangle^2 \rightarrow D$ and $b'(s) := \pi_s a'(s)$. We prove the following result using the probabilistic method.

Lemma 29 For large enough n , the functions a, a' above can be chosen in such a way that $\text{Im } a \cup \text{Im } b \cup \text{Im } a' \cup \text{Im } b'$ has exactly $4 \binom{n}{2}$ elements.

Proof. Let $T(s)$ denote the set $\{a(s), b(s), a'(s), b'(s)\}$, for any $s \in \langle n \rangle^2$. The objective is to choose a and a' so that

$$\text{for every } s = \langle i, j \rangle \in \langle n \rangle^2: |T(s)| = 4, \quad (15)$$

and so that

$$\text{for every } s, r \in \langle n \rangle^2, s \neq r: T(s) \cap T(r) = \emptyset. \quad (16)$$

For $s = \langle i, j \rangle \in \langle n \rangle^2$, define D_s as the set of all $x \in D$ such that $x_i > x_j$. Let q_n be the size of this set. Note that q_n does not depend of the particular values of $i < j$.

If $a(s), a'(s) \in D_s$, then an equivalent condition for (15) is that $a(s) \neq a'(s)$. To see the sufficiency of this condition (the necessity being obvious), note that if $a(s) \neq a'(s)$,

then necessarily $b(s) \neq b'(s)$ by construction, and that also $b(s), b'(s) \notin D_s$, so that $a(s) \neq b(s), b'(s)$; similarly $a'(s) \neq b(s), b'(s)$.

We will prove this lemma by making a simple use of the probabilistic method. Suppose that for every fixed s , $a(s)$ is a discrete random variable that takes any value in D_s with equal probability. Let $a'(s)$ be similarly defined, and in such a way that all random values of both variables for any $s, r \in \langle n \rangle^2$ are independently distributed. We show that the probability that both (15) and (16) hold is greater than zero, which implies the result.

For example, for any given s , we compute the probability that $a(s) = a'(s)$:

$$P(a(s) = a'(s)) = \sum_{x \in D_s} P(a(s) = x, a'(s) = x) = q_n \frac{1}{q_n} \frac{1}{q_n} = \frac{1}{q_n}.$$

If $r \neq s$, we compute the probability that $a(s) = a(r)$:

$$P(a(s) = a(r)) = \sum_{x \in D_s \cap D_r} P(a(s) = x, a(r) = x) = |D_s \cap D_r| \frac{1}{q_n} \frac{1}{q_n} \leq \frac{1}{q_n}.$$

This bound also holds for $a(s) = a'(r)$, and for any other combination of the functions a, a', b, b' (note that it is quite possible that, say, $a(s) = b(r)$ for $r \neq s$). Thus we obtain for $s \neq r$ the bound:

$$P(T(s) \cap T(r) \neq \emptyset) \leq 16 \frac{1}{q_n}.$$

Now the event that either (15) or (16) fails can be described as

$$\begin{aligned} & P(\text{either (15) or (16) fails}) \\ & \leq \sum_{s \in \langle n \rangle^2} P(a(s) = a'(s)) + \sum_{r, s \in \langle n \rangle^2, r \neq s} P(T(s) \cap T(r) \neq \emptyset) \\ & \leq \sum_{s \in \langle n \rangle^2} 16 \frac{1}{q_n} + \sum_{r, s \in \langle n \rangle^2, r \neq s} 16 \frac{1}{q_n} = \binom{n}{2} \left(\binom{n}{2} + 1 \right) 16 \frac{1}{q_n} \end{aligned} \tag{17}$$

We give a lower bound for q_n as follows. Consider the set D' of vectors $x \in \{0, \dots, p-1\}^{n-2}$ such that $S(x) = \lfloor (n-2)(p-1)/2 \rfloor$, which has cardinality $d_{n-2,p}$. A straightforward injection of D' into $D_{(n-1,n)}$ is given by the function $[x_1, \dots, x_{n-2}] \rightarrow [x_1, \dots, x_{n-2}, p-1, 0]$ (note $\lfloor (n-2)(p-1)/2 \rfloor + (p-1) = \lfloor \frac{n(p-1)}{2} \rfloor$).

This implies that

$$q_n \geq d_{n-2,p} \geq \frac{p^{n-3}}{n},$$

by Lemma 5. Since $\binom{n}{2} \left(\binom{n}{2} + 1 \right) \frac{16}{q_n} \leq \binom{n}{2} \left(\binom{n}{2} + 1 \right) \frac{16n}{p^{n-3}} < 1$ for large enough n , the statement follows from (17). \blacksquare

Now consider functions $a, a' : \langle n \rangle^2 \rightarrow D$ as in the statement of Lemma 29, and their associated functions b, b' such that the conclusion of this Lemma is satisfied. In particular, for every $s = \langle i, j \rangle$, $a(s)_i \neq a(s)_j$ must hold (else $a(s) = b(s)$). By swapping the values of $a(s)$ and $b(s)$ if necessary, we can assume without loss of generality that $a(s)_i > a(s)_j$, $b(s)_i < b(s)_j$ for every s . Repeat the same procedure with a', b' .

Define $c, c' : < n >^2 \rightarrow \Pi$ by $c(s) = \min(a(s), b(s))$, $c'(s) = \min(a'(s), b'(s))$. In particular $c(s) < a(s)$, $c(s) < b(s)$ for every s , and similarly with c' .

Define the values of γ in such a way that $\gamma(a(s)) = b'(s)$, $\gamma(b(s)) = a'(s)$ for every s . This is possible by Lemma 29, since the states $a(s)$, $b(s)$ are all different from each other. Since we are still free to choose the values $\gamma(a'(s))$, $\gamma(b'(s))$ for every s (again by Lemma 29), we can define the values of γ in such a way that γ generates a single orbit of period $|D|$.

Let $g : \Pi \rightarrow \Pi$ be a cooperative extension of γ . For any fixed $s = < i, j >$, since $c(s) \leq b(s)$, it follows by cooperativity $g(c(s)) \leq g(b(s))$. But then

$$g(c(s))_j \leq g(b(s))_j = a'(s)_j < b'(s)_j = g(a(s))_j.$$

Since $c(s)_k = a(s)_k$ for all $k \neq i$, and $c(s)_i < a(s)_i$, the equation above shows that $< i, j > \in A_{a(s)}^*$. Similarly, using $c(s) \leq a(s)$,

$$g(c(s))_i \leq g(a(s))_i = b'(s)_i < a'(s)_i = g(b(s))_i,$$

which in the same way implies $< j, i > \in A_{a(s)}^*$. This shows that $\bigcup_{x \in D} A_x^*$ contains all arcs $< i, j >$ with $i \neq j$; in particular, the system (Π, g) is weakly irreducible along D .

Finally, assume that g is the Smale extension of γ as defined in Section 4. Note that for every s , the only elements $d \in D$ with $c(s) \leq d$ are $a(s)$ and $b(s)$. Thus $g(c(s)) = \inf\{a'(s), b'(s)\}$, and our choice of these values implies that $(g(c(< i, j >)))_k = (g(a(< i, j >)))_k$ for all $k \neq i, j$. Similarly, if $e(s) = \max\{a(s), b(s)\}$, then $(g(c(< i, j >)))_k = (g(a(< i, j >)))_k$ for all $k \neq i, j$. Thus $A_{a(< i, j >)}$ contains no arcs from the set $\{i, j\}$ into its complement, and thus $G_{a(< i, j >)}$ cannot be strongly connected when $n > 2$. It follows that (Π, g) cannot be irreducible along D . \blacksquare

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